Seminar on Hopf algebras (BSc and MSc)

Ingo Runkel

The seminar talks are 90 minutes each, so it is recommended to plan them for **70-80 minutes** to allow for questions and comments. You are asked to prepare a **handout** of 2-3 pages on your topic and present this to me **2 weeks before** the seminar. We will meet a few days after I receive the handout to discuss the content of your talk.

The outlines state which material should be covered. For time reasons it is usually not possible to include all proofs – please choose what you present with the aim to make the seminar **comprehensible and instructive for the audience**. The references given should cover most aspects of the seminar, possibly multiple times.

1 Foundations

F1) Algebras

Definition of algebra over a field k. Examples: function algebra; matrix algebra; group algebra (also works for monoid); field extensions; Sweedler's 4-dimensional Hopf algebra (only algebra part). Tensor algebra and its universal property, basis of tensor algebra. Tensor product of algebras. Algebra homomorphisms; examples (e.g. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[X]/(X^2+1) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[X]/(X^2-1)$). Two-sided ideals; fundamental homomorphism theorem for algebras (same as for rings).

[La, III §1, XVI §6 §7], [Bu, Sec. 9] [Ka, Sec. II.4, II.5], [Sw Sec. 3.2] [DNR, Sec. 4.3]

F2) Coalgebras

Algebra axioms as commuting diagrams. Coalgebra axioms as commuting diagrams. The embedding $U^* \otimes V^* \to (U \otimes V)^*$; the vector space-dual of a coalgebra is an algebra; the dual of a finite-dimensional algebra is a coalgebra. Examples: the field k, coalgebra of a set (dual to algebra of functions), dual group algebra, matrix coalgebra. Tensor product of coalgebras. Homomorphisms of algebras and of coalgebras via diagrams (to see they are dual to each other). Coideals (two-sided only), fundamental homomorphism theorem for coalgebras.

[Sw, Sec. 1.1–1.3, 3.2] [Sw, Thm. 1.4.7] [Ka, Sec. III.1, Prop. III.1.2, III.1.3]

F3) Bialgebras

Sweedler's sigma notation for coproduct and iterated coproduct; some defining identities via Sweedler notation. Algebra-map and coalgebra-map characterisation of bialgebras are equivalent. The dual of a finite-dimensional bialgebra is again a bialgebra. Bialgebras H^{op}, H^{cop}, $H^{op,cop}$. Commutativity and co-commutativity. Example: group bialgebra $k[G]$ (also works for a monoid), function bialgebra, the group and function bialgebras are each others duals. Non-example: the algebra $M_n(k)$ of $n \times n$ matrices does not allow for a bialgebra structure. Example: tensor algebra (without explicit shuffle expression for coproduct – maybe degree 2 as an example).

[Sw, Notation in Sec. 1.1, Prop. 3.1.1] [Ka, Not. III.1.6, Sec. III.2, Thm. III.2.1, Thm. III.2.4] [Ma, Ex. 1.5.5] [DNR, Ex. 4.1.9 (and solution later on) Sec. 4.3]

F4+F5) Hopf algebras, part 1+2

For an algebra A and a coalgebra C, $\mathrm{Hom}(C, A)$ is an algebra (convolution algebra); compare $\mathrm{Hom}(C, k)$ to the dual of a coalgebra. Definition of an antipode for a bialgebra via convolution, uniqueness. Characterisation via $\sum_{(x)} x' S(x'') = \varepsilon(x)1 = \sum_{(x)} S(x')x''$. Definition of a Hopf algebra. The antipode is an anti-(co)algebra map; $S^2 = id$ for commutative or co-commutative Hopf algebras. The dual of a finite-dimensional Hopf algebra is a Hopf algebra. Examples: group algebra $k(G)$ and function algebra kG, Sweedler's 4-dimensional Hopf algebra, tensor algebra. A bialgebra map between two Hopf algebras automatically preserves the antipode. Hopf ideals, fundamental homomorphism theorem for Hopf algebras. Definition of group-like elements, they form a monoid (for a bialgebra) / a group (for a Hopf algebra). Example: the group-like elements of $k[G]$. Primitive elements, bialgebra (Hopf algebra) map $T(V) \to H$, where $V \subset H$ are the primitive elements. Maybe: a finite-dimensional Hopf algebra over a field of characteristic zero does not contain non-zero primitive elements.

[Sw, Sec. 4.0, Prop. 4.0.1, Thm. 4.3.1] [Ka, Sec. III.3, Prop. III.2.6, Prop. III.2.7, Prop. III.3.3, Thm. III.3.4, Lem. III.3.6, Prop. III.3.7] [DNR, Prop. 4.2.13, Ex. 4.2.16 (and solution later on), Sec. 4.3]

F6) Modules, tensor products and duals

Representation of an algebra, module over an algebra. Homomorphism of modules. Action of a bialgebra on the tensor product of two modules. Example: group algebra $k[G]$, function algebra kG (give all one-dimensional modules, compute their tensor product). Associativity of the tensor product: $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$. The trivial module is a unit for $\otimes$. Commutativity of $\otimes$ for cocommutative bialgebras. Example: $\otimes$ not commutative for kG with G non-commutative. For Hopf algebras have module structure on $\mathrm{Hom}(U, V)$, the dual module V^*, evaluation and coevaluation maps.

[Pi, Sec. 5.5], [La, III §1], [Ka, Sec. I.1], [Ka, Sec. III.5, Prop. III.5.3 a,b)]

2 Quasitriangular Hopf algebras

Q1) R-matrices and Yang-Baxter equation

Quasi-cocommutative bialgebra (Hopf algebra), quasi-triangular (or braided) bialgebra (Hopf algebra). Examples: cocommutative Hopf algebra; Sweedler's Hopf algebra. The Yang-Baxter equation for a linear automorphism $c : V \otimes V \to V \otimes V$, write in terms of a basis. Properties of R-matrix: Yang-Baxter, counit, antipode, inverse of R. The R-matrix of a bialgebra H gives an intertwiner $U \otimes V \to V \otimes U$ of H-modules. The R-matrix gives a solution to the Yang-Baxter equation for every module.

[Ka, Sec. VIII.1, Sec. VIII.2, Thm. VIII.2.4, Sec. VIII.3, Prop. VIII.3.1]

Q2) The quantum double

Skew-pairing of bialgebras A, B. Bialgebra structure on $B \otimes A$ for skew-paired bialgebras. Hopf algebra structure on $B \otimes A$. The quantum double of a Hopf algebra, explicit form of structure maps. Example: The quantum double of a group algebra. The quantum double is quasi-triangular.

[KS, Sec. 8.2, Thm. 9 in Sec. 8.2.2] [Ka, Sec. IX.4.3]

Q3) Yang-Baxter equation and Artin braid group

The Artin braid group B_n on n strands; generators and relations; geometric presentation; equivalence of the two (without proof). Surjection to S_n. A solution of the Yang-Baxter equation gives representations of B_n for each n. Examples of representations of B_n that do not factor through S_n originating from Hopf algebras.

[Ka, Sec. X.6], [KRT, Sec. 1, Sec. 2.3], [KT, Ch. 1].

3 The quantum group $U_q(sl(2))$

M1+M2) Enveloping algebras of Lie algebras and $U_q(sl_2)$, parts 1+2

Definition of a Lie algebra. Example: matrix algebras with commutator (i.e. $gl(n)$), trace-less matrices (i.e. $sl(n)$), $sl(2)$ in terms of generators E, F, H. Universal enveloping algebra $U(L)$ of a Lie algebra L and its universal property. The Poincaré-Birkhoff-Witt theorem and the resulting basis of $U(L)$. $U(L)$ is a Hopf algebra (maybe without the shuffle-product representation of the coproduct); L are primitive elements in $U(L)$. q-numbers. Definition of $U_q(sl(2))$ as an algebra via generators E, F, K. Basis of $U_q(sl(2))$ (without proof). Relation to $U(sl(2))$ via a different presentation of $U_q(sl(2))$ (as in [Ka, Sec. VI.2]), and informally by recovering $U(sl(2))$ relations as $q \to 1$ limits (with $K = q^H$, again informally). Hopf algebra structure on $U_q(sl(2))$; the square of the antipode is inner.

[Ka, Sec. V.1–V.3, Thm. V.2.1, Prop. V.2.4, Sec. VI.1–VI.2, Sec. VII.1, Prop. VIII.1.1, Def. VI.5.6, Prop. VI.5.8,

M3) R-matrix for a quotient of $U_q(sl(2))$

A finite-dimensional quotient u_q of the Hopf algebra $U_q(sl(2))$ (with some sketch of the proof). The Hopf algebra B_q; some sketch of Hopf algebra structure on $(B_q^{op})^*$. The surjection $D(B_q) \to u_q$ (without proof); u_q is quasi-triangular. R-matrix for u_q in terms of E, F, K (explain what computation need to be done, skip details).

The two-dimensional irreducible representation V_1 of u_q; evaluation of R-matrix on $V_1 \otimes V_1$; resulting solution of the Yang-Baxter equation.

[Ka, Def. VI.5.6, Prop. VI.5.8, Sec. IX.6, Prop. IX.6.1, Prop. IX.6.2, Cor. IX.6.7, Thm. IX.7.1, Eqn. (3.1)–(3.3) in Sec. VI.3 (specialised to 2-dim. repn.), Application IX.7.4]

4 Reconstruction (2 seminars)

Categories, functors, natural transformations. Examples: the category of vector spaces over a field k, category of modules over an algebra. Reconstruction of an algebra from its category of modules and a functor to vector spaces. Monoidal (aka tensor) categories, monoidal functors. Examples: the category of vector space, the category $\mathrm{Rep}(H)$ for a bialgebra H. Dual objects in a monoidal category. Example: $\mathrm{Rep}(H)$ for a Hopf algebra H. Twisting of the coproduct in a Hopf algebra; twisted Hopf algebras give equivalent monoidal categories. (Time permitting: definition of braided monoidal categories and example from quasi-triangular Hopf algebras.) Given a monoidal category and a fibre functor (to vector spaces), one can construct a bialgebra. For a rigid category, one constructs a Hopf algebra. (Time permitting: for a braided category, one constructs a quasi-triangular Hopf algebra.

[Ma, Sec. 9.1, (9.2), 9.3], [CP, Sec. 4.2.E, Examples 5.1.3–5], [Ma, 9.4.1, Exercise 9.4.4], [CP, Sec. 5.1.E]

Quellen

[Bu] Bump, Lie groups

[CP] Chari, Pressley, Guide To Quantum Groups

[DNR] Dascalescu, Nastasescu, Raianu, Hopf algebras – an introduction

[Ka] Kassel, Quantum groups

[KRT] Kassel, Rosso, Turaev, Quantum groups and knot invariants

[KS] Klimyk, Schmudgen, Quantum groups and their representations

[KT] Kassel, Turaev, Braid Groups

[La] Lang, Algebra

[Ma] Majid, Foundations of Quantum Group Theory

[Pi] Pierce, Associative algebras

[Sw] Sweedler, Hopf algebras